Using the Stochastic Variational Method in Momentum Representation to Solve Nuclear Few-Body Problems

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by

Thomas J. Colvin

The Ohio State University

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Project Advisor: Professor R.J. Furnstahl, Department of Physics

Abstract

The Stochastic Variational Method (SVM) is a powerful way of numerically calculating bound-state energies for few-body systems. It has, however, never been used in momentum representation. Recent advances using Similarity Renormalization Group (SRG) methods allow inter-nucleon potentials to be evolved, in momentum representation, into a form that may be suitable for calculations using variational techniques. Using the deuteron ($N = 2$) as a simple model, the SVM is shown to work in momentum representation as a proof-of-principle. Further, the SVM is also shown to be effective on an SRG-evolved potential. These results motivate the generalization of the SVM to $N > 2$ bodies for use on SRG evolved potentials and point the way towards that goal.

Introduction

Typically, inter-nucleon interactions are highly non-perturbative, such that variational calculations can become prohibitively complex. This is primarily caused by the strong short-range repulsive and tensor forces of conventional nucleon-nucleon potentials. Due to the strong repulsion, it is highly unlikely that any two nucleons will be found near to each other. Thus, to treat such potentials variationally requires highly correlated and complex trial wave functions that take this insight into account. These problematic short-range interactions are, however, also highly dependent upon the resolution at which you examine them. Thinking in terms of momentum, the short-range interactions are described by high momenta because they would require short wavelengths to properly resolve their sharply changing behaviour. An interaction that exhibits this high momentum behavior is said to be "hard".

In momentum representation, inter-nucleon potentials that have a variable momentum cutoff can be used to simplify the interactions by removing the "hard" elements as the cutoff is lowered. These interactions can then be drastically simplified by using the renormalization group to lower the momentum cutoff for an internucleon potential while preserving all observables. This technique produces a "soft" potential that can be described using only

low momenta because the harder aspects have been removed. These soft potentials with low momentum cutoffs are generically known as 'Vlowk'. It has been found that methods which were previously limited or unsuccessful using hard potentials seem to be effective using our newly constructed soft potentials. One such method that I have been revisiting for use with Vlowk is the Stochastic Variational Method, which has previously been applied only in coordinate representation.

The Stochastic Variational Method (SVM) is a powerful way of numerically calculating bound state energies for few-body systems. It is also very easy to understand with some basic knowledge of quantum mechanics. In fact, the entire process can be well understood by knowing only the following two theorems:

1. *Let H be a Hermitian operator with discrete eigenvalues $E_1 \leq E_2 \leq \ldots$ and let $\epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_K$ be the eigenvalues of H restricted to the subspace ν_K of a linearly independent set of K functions $\psi(\alpha_1), \psi(\alpha_2), \ldots, \psi(\alpha_K)$. Then*

$$E_1 \leq \epsilon_1, E_2 \leq \epsilon_2, \ldots, E_K \leq \epsilon_K.^{[1]}$$

2. *Let $\epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_K$ be the eigenvalues of a Hermitian operator H restricted to the subspace ν_K of linear combinations of independent functions $\psi(\alpha_1), \psi(\alpha_2), \ldots, \psi(\alpha_K)$. Let $\epsilon'_1 \leq \epsilon'_2 \leq \ldots \leq \epsilon'_{K+1}$ be the eigenvalues of H restricted to the subspace ν_{K+1} of linear combinations of independent functions $\psi(\alpha_1), \psi(\alpha_2), \ldots, \psi(\epsilon_K)$, and $\psi(\alpha_{K+1})$. Then*

$$\epsilon'_1 \leq \epsilon_1 \leq \epsilon'_2 \leq \epsilon_2 \leq \ldots \leq \epsilon'_K \leq \epsilon_K \leq \epsilon'_{K+1}.^{[1]}$$

Theorem 1 is commonly called 'the variational principle'. It states that for a finite dimensional representation of a system, the eigenvalues that are obtained will provide upper bounds to the actual eigenvalues. Thus, minimizing the eigenvalues in ν_K is equivalent to improving their agreement with the real values. For my current interests I am focusing only on finding the lowest eigenvalue, however the variational principle states that *every* calculated ϵ_i is an upper bound to the corresponding exact E_i. In application, estimates for the exact eigenvalues of a system can be found by constructing a basis and computing

the eigenvalues of the hamiltonian in this basis. The only real restriction on these functions is that they be linearly independent. To obtain *good* estimates, the subspace ν_K that they span must also have a meaningful overlap with the exact eigenfunctions.

Theorem 2 states that for a given finite dimensional basis, if a single new basis state is added, then the new eigenvalues cannot be worse than the corresponding eigenvalues that were obtained before. This is the crux of the SVM. By adding a new basis function, no matter what it is, the eigenvalues can only stay the same or get better. Qualitatively, this is because if a new basis function is added that has no overlap whatsoever with one of the exact eigenfunctions, then it will not get used. Being unused, it contributes nothing to the computed eigenvalues. If, however, the function overlaps with a significant piece of the exact eigenfunctions, then its inclusion improves the restricted representation of the system and thus certainly brings the calculated eigenvalues into closer agreement with the exact values.

Standing upon these two theorems, the SVM is conceptually clear. Starting with a basis function that depends on one or more parameters, an equation for the ground state is found and numerically minimized through stochastic variation of the parameters. This is where the parameters to be minimized are repeatedly chosen at random and then computed to determine which of them gave the best answer. Stochastic, as opposed to deterministic, minimizations are useful because they are faster and less likely to become trapped in local minima. Theorem 1 assures that the computed eigenvalue will be an upper-bound on the true eigenvalue of the system. Next, another basis function is added and similarly minimized. The new ground state energy obtained from the two-dimensional basis will most likely be less than, or at least equal to, the one-dimensional basis by Theorem 2. The computed ground state energy is then driven downwards by the similar addition of further basis states.

The trial wavefunctions used for the SVM technique are chosen to be correlated gaussians because they meet the following criteria:

- Their generalization to N-body systems should be easy.
- They yield analytically calculable matrix elements.
- They can be easily adapted to the system's permutational symmetry.

- They are able to approximate rapidly changing functions.[1]

In summary, the basic steps involved in building up a basis using the SVM with the appropriate trial wave functions are:

1. Generate a pool of N random parameters to be used for the minimization.
2. For the current basis, calculate the ground state energy of the system using each of the N parameters.
3. Keep the parameter that produced the lowest ground state energy to be the current basis parameter.
4. Add another basis function and repeat.[1]

After having built a finite-dimensional basis, it is possible to minimize the computed ground state energy further by refining the basis functions that have already been selected. Each basis function is individually cycled through and again stochastically minimized such that it yields the lowest possible ground state energy. This process can be repeated as often as desired or until the eigenvalue can be minimized no further:

1. Generate a pool of N random parameters.
2. Holding the other basis functions constant, calculate the binding energy using each of the new N parameters
3. Keep the parameter that yields the lowest energy. This may be the original parameter.
4. Move to the next basis function and repeat.[1]

The combination of these two methods, building and then refining a basis to minimize the eigenvalues of a system, constitute the heart of the SVM. It has been previously applied to various nuclear few-body problems in coordinate representation with success, however, it has not been applied in momentum representation due to the extreme complexity required

to handle the hard pieces of nuclear potentials. Attempts in the 1960's and 1970's to produce meaningful soft potentials failed because they led to nuclear collapse instead of bound states. The source of the failure was the inconsistent and incomplete treatment of the three-body forces, which are now known to be very important in an accurate description of nuclei. Recent developments with Similarity Renormalization Group techniques have been used to revisit the problem of soft potentials and are producing promising results. Thus, a method for variationally treating these new potentials is motivated.

The Deuteron

Given that the SVM has never been applied in momentum representation before, it is necessary to first show that the process works. This was done by examining the deuteron (a proton and a neutron) in a simple three-dimensional case. First, I modelled the system in coordinate representation with a Malfliet-Tjon potential (MTV). The MTV is a sum of two Yukawas, one attractive and one repulsive:

$$MTV(r) = \frac{V_r}{r} e^{-d_r r} - \frac{V_a}{r} e^{-d_a r} \ . \tag{1}$$

Since the MTV is a central potential, the deuteron reduces to a one-dimensional radial problem. The parameters V_r, V_a, d_r, and d_a are empirically determined to fit the form of the potential to the experimental data.

Table 1: Experimentally determined parameters for the MTV with binding energy -2.2307 MeV [2]

Parameter	Repulsive	Attractive	Units
V	1438.7228	626.8932	MeV $\cdot$ fm
d	3.11	1.55	fm^{-1}

To change this potential into momentum representation, I applied a Fourier transformation. As noted above, Equation 1 is a sum of two Yukawas, which have the general

form

$$\langle \vec{r} \,|V|\, \vec{r}\,' \rangle = V(\vec{r}) \langle \vec{r} \,|\, \vec{r}\,' \rangle = \frac{V_0}{r} e^{-d_0 r} \delta^3(\vec{r} - \vec{r}\,') \; . \tag{2}$$

Written in momentum representation, this becomes the fourier transform

$$\langle \vec{k} \,|V|\, \vec{k}\,' \rangle = \int d^3\vec{r} \int d^3\vec{r}\,' \langle \vec{k} \,|\, \vec{r} \rangle \langle \vec{r} \,|V|\, \vec{r}\,' \rangle \langle \vec{r}\,' \,|\, \vec{k}\,' \rangle \tag{3}$$

where

$$\langle \vec{r} \,|\, \vec{k} \rangle = \left(\frac{1}{2\pi} \right)^{\frac{3}{2}} e^{i\,\vec{k}\,\cdot\,\vec{r}} \; . \tag{4}$$

The plane waves of Equation (4) can be decomposed into a infinite sums of their constituent "partial waves" [4]

$$e^{i\,\vec{k}\,\cdot\,\vec{r}} = 4\pi \sum_{lm} i^l j_l(kr)\, Y_l^{m^*}(\hat{k})\, Y_l^m(\hat{r}) \; . \tag{5}$$

After substituting this expansion into the Equation (3) and performing the trivial integral on $\vec{r}\,'$ over the delta function in the potential, it becomes

$$\langle \vec{k} \,|V|\, \vec{k}\,' \rangle = \frac{2}{\pi} \sum_{lml'm'} \int d^3\vec{r} j_l(kr)\, Y_l^m(\hat{k})\, Y_l^{m^*}(\hat{r}) V(r) j_{l'}(k'r)\, Y_{l'}^{m'^*}(\hat{k}')\, Y_{l'}^{m'}(\hat{r}) \; . \tag{6}$$

Modeling the deuteron with the MTV requires only the s-wave piece of the partial waves. The s-wave is characterized by $l = 0$ and $m = 0$. Performing the angular integral over the orthonormal spherical harmonic functions and applying the s-wave approximation simplifies Equation (6) to

$$\langle \vec{k} \,|V|\, \vec{k}\,' \rangle = \frac{1}{2\pi^2} \int_0^\infty \frac{Sin(kr)}{kr} \frac{V_0}{r} e^{-d_0 r} \frac{Sin(k'r)}{k'r} r^2 dr \; . \tag{7}$$

Performing this final integration with Mathematica and noting that the MTV is a sum of two yukawas, the s-wave approximation to the deuteron in momentum representation is found to be

$$MTV(k,k') = \frac{1}{4\,k\,k'}\left[V_r\,\ln\left(\frac{(k'+k)^2+d_r^2}{(k'-k)^2+d_r^2}\right) - V_a\,\ln\left(\frac{(k'+k)^2+d_a^2}{(k'-k)^2+d_a^2}\right)\right]\,. \tag{8}$$

With this model, I choose to use a basis of gaussian wave-functions

$$\langle k|\psi(b)\rangle = \left(\frac{b}{\sqrt{\pi}}\right)^{\frac{3}{2}} e^{-\frac{1}{2}b^2k^2}\,, \tag{9}$$

each containing a single unique width parameter and a relative momentum variable. The trial wave functions are Gaussians because they are easy to integrate and can often yield closed forms which enable purely analytic calculations. These closed forms will be crucial in the future when we move to systems with more bodies, such as the triton (proton and two neutrons).

Using the correlated Gaussian basis functions, Figure 1 shows the SVM approximating the MTV binding energy. This is a clear demonstration that the SVM works in momentum representation because the calculated binding energy quickly converges to the expected value of -2.2307 MeV. As predicted by Theorem 2, the addition of new basis functions never yields an increase in the binding energy calculation. The prediction of Theorem 1 is also seen to hold because the calculated binding energy never drops below the true value. Instead, it bounds it from above. It is also apparent that the calculation is not getting stuck in a local minimum, as is expected of a stochastic minimization. To the knowledge of our group, this marks the first time that the SVM has ever been used to do a calculation in momentum representation.

The refinement process is also seen to be successful in Figure 2. The first refinement pass makes significant improvements to the calculated binding energy. With the refinement of the fifth basis function, the energy has nearly converged upon its final value. The refinement on the sixth basis function was unable to find a lower energy and thus it remains the same. The

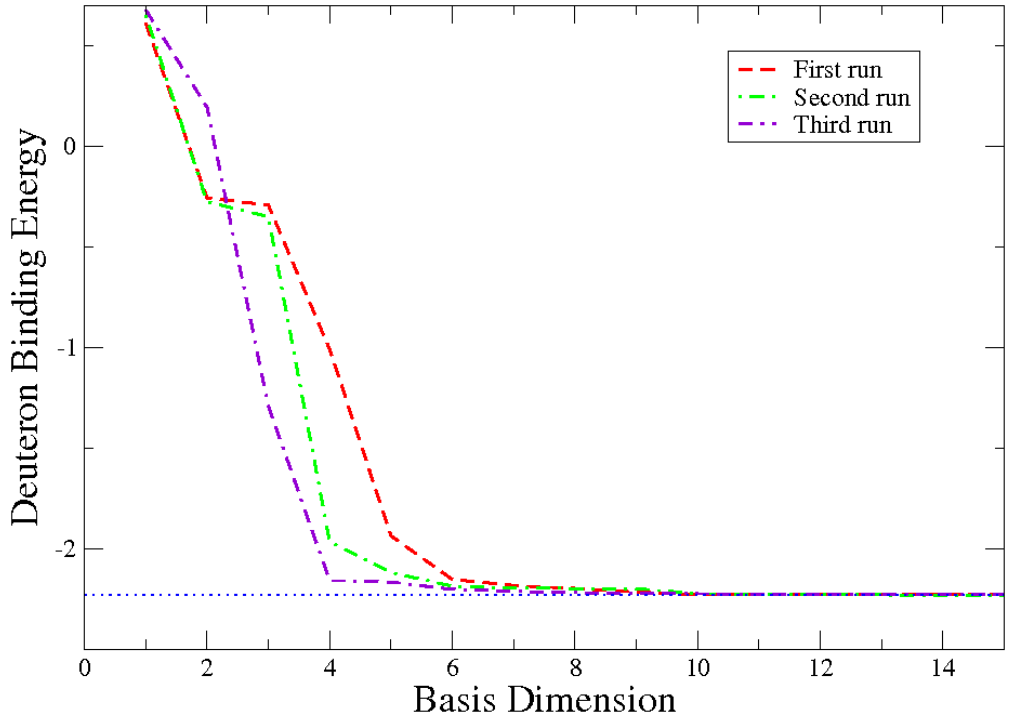

Figure 1: Calculated deuteron binding energy as basis functions are added for three independent basis constructions using the MTV. The expected energy, -2.2307 MeV, is labeled with a straight dotted line.

following two refinement passes are largely unable to reduce the energy as well. This is due to the fact that the existing basis functions are doing such a good job of representing the ground state eigenfunction that it is difficult to randomly find a new basis function that accounts for the missing pieces without sacrificing in other more important areas. Additionally, both of the basic SVM theorems are seen to hold because the calculated energy never gets worse and always bounds the true energy from above.

In theory, this process will always work, but in a computer things can go wrong. The main problem is the seeming loss of linear independence in the gaussian basis. As the basis grows, it is possible for one of the functions to become nearly a linear combination of the others. This near dependence causes one of the eigenstates to have a very small eigenvalue, leading it to have large expansion coefficients in the gaussian basis. These large coefficients amplify any errors that may be present in the calculation. This effect has been seen to lead to spurious deep bound states at large basis sizes. In general, the problem can be

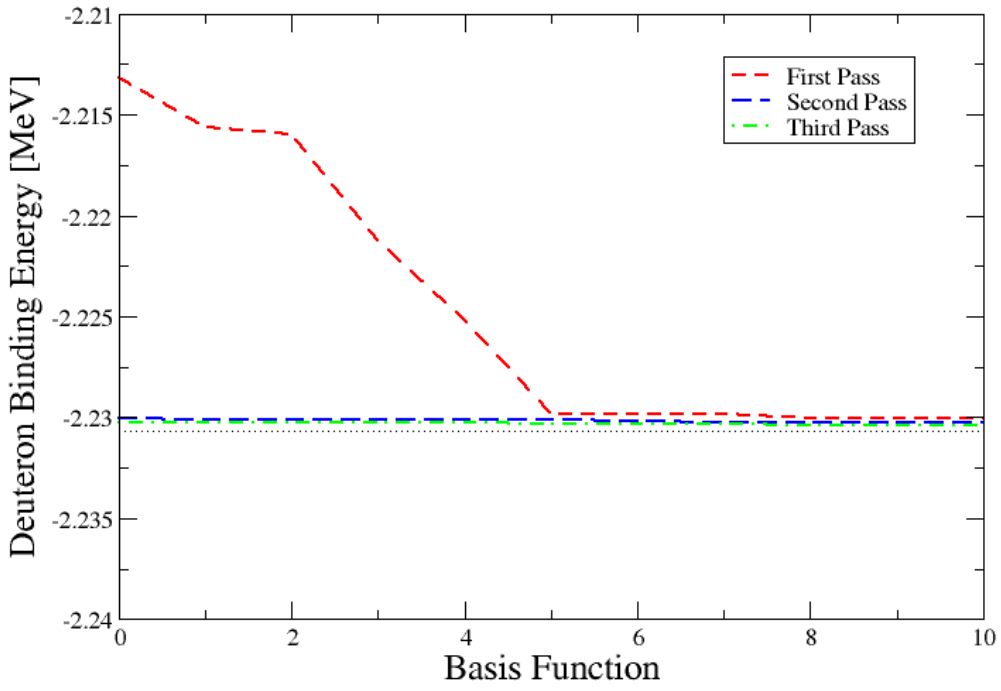

Figure 2: Calculated deuteron binding energy as individual basis functions are refined for three consecutive passes through the MTV potential on a pre-constructed basis size of 10. The expected energy is again labeled with a straight dotted line.

alleviated by constructing an orthonormal basis from the gaussians and insisting that all expansion coefficients fall below a certian threshold. By using only gaussians that conform to this requirement, a proper degree of linear independence can be preserved. For my specific purpose, however, the eigenvalue converges quickly and makes large basis sizes unnecessary. Thus the problem is easily avoided.

Similarity Renormalization Group Potentials

Similarity Renormalization Group (SRG) transformations are a way to decouple the low-momentum from the high-momentum physics. They rely on unitary transformations to drive the hamiltonian matrix into a band-diagonal form. The absence of strong off-diagonal elements means that there is no longer a coupling between the high- and low-momentum physics. By suppressing the off-diagonal matrix elements with unitary transformations, the

observables of the system are unchanged at *all* energies. Thus, to obtain a soft potential, the high momentum pieces of an SRG evolved potential can be effectively disregarded because they no longer affect the low momentum observables, which were themselves unchanged by the transformation. The parameter λ provides a measure of the off-diagonal matrix element strengths and approximately defines the low-momentum boundary. Decreasing λ corresponds to a more diagonalized hamiltonian and fewer matrix elements required to properly calculate low-momentum observables.[3]

Having already demonstrated that the SVM works in momentum representation, it is also of interest to apply it to SRG evolved soft potentials. A useful model potential to consider is that of a two-particle simple harmonic oscillator with a one-dimensional boson gaussian

$$V(x) = -\frac{g}{b\sqrt{\pi}} e^{-(x/b)^2} \ , \tag{10}$$

where g and b are parameters that determine the size and shape of the well. In momentum representation, this potential becomes

$$V(k,k') = -\frac{g}{2\pi} e^{-\frac{b^2}{4}(k-k')^2} \ . \tag{11}$$

For given values of the two parameters, the resulting potential can be SRG evolved. Figure 3 shows an example of the bare potential and the same potential as it is SRG evolved. Initially there are strong off-diagonal elements in the bare potential, which indicate a coupling of low- to high-momentum. This can be seen by picking a relatively small value of k and noticing that there is a significantly higher value of k' that has a non-zero contribution to the total potential. This property can be seen to decrease as the momentum cutoff λ is lowered and the matrix approaches a band-diagonal form. After the evolution, only comparable values of k and k' contribute to the total potential and thus the high- and low-momentum physics have been decoupled.

Using the SRG evolved potential with a cutoff of $\lambda = 20.0$, Figure 4 compares the performance of the SVM on both the evolved and bare potential. The SRG evolved potential,

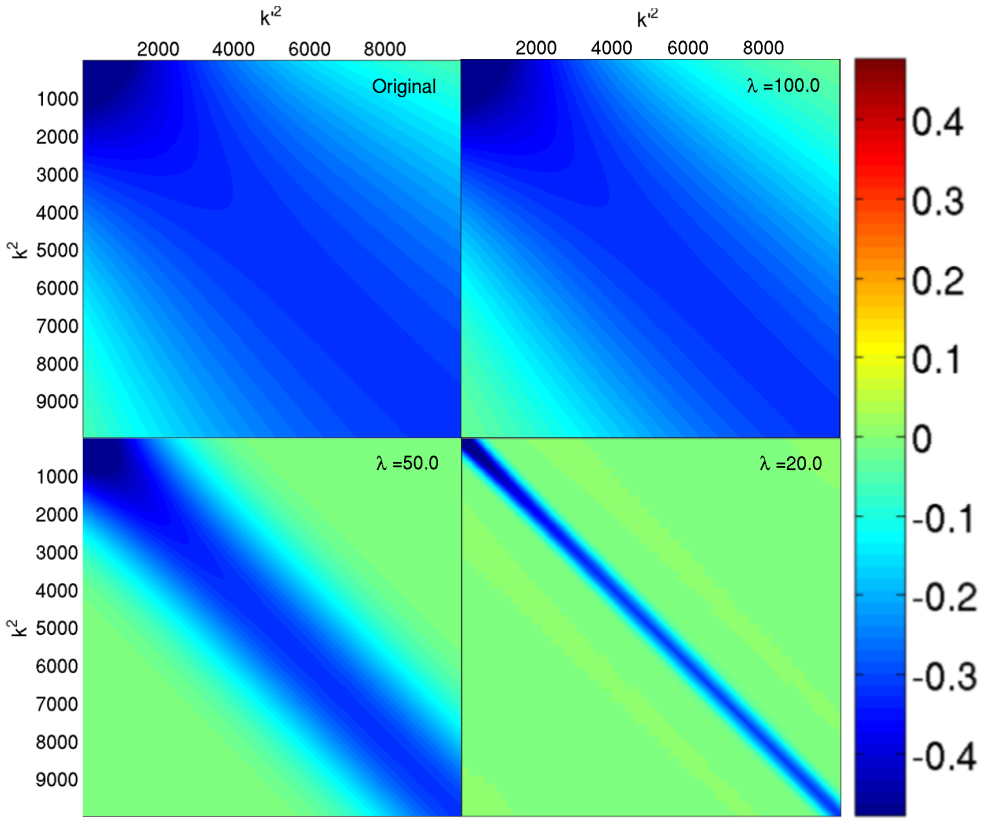

Figure 3: SRG evolution of the one-dimensional simple harmonic oscillator potential with $g = 2.0$ and $b = 0.030$ down to $\lambda = 20.0$.

shown on the left, yields the expected binding energy in all three independent runs, while the bare potential, shown on the right, manages to miss during the second run. This is ultimately not a problem because the second run would surely converge upon the correct value after a basis refinement. It does, however, help to illustrate that even though the SVM converges upon the correct answer for both potentials, it does so *faster* with the SRG evolved potential. This is due to the SRG potential being softer than the bare, as can be easily seen in Figure 3. The absence of high-momentum elements in the potential simplifies the eigenfunction from having to accommodate them, thus making it easier to represent in a restricted basis and to calculate variationally.

The physical interpretation of λ is the boundary that defines the low-momentum region, which has been decoupled from the high-momentum pieces of the potential without altering

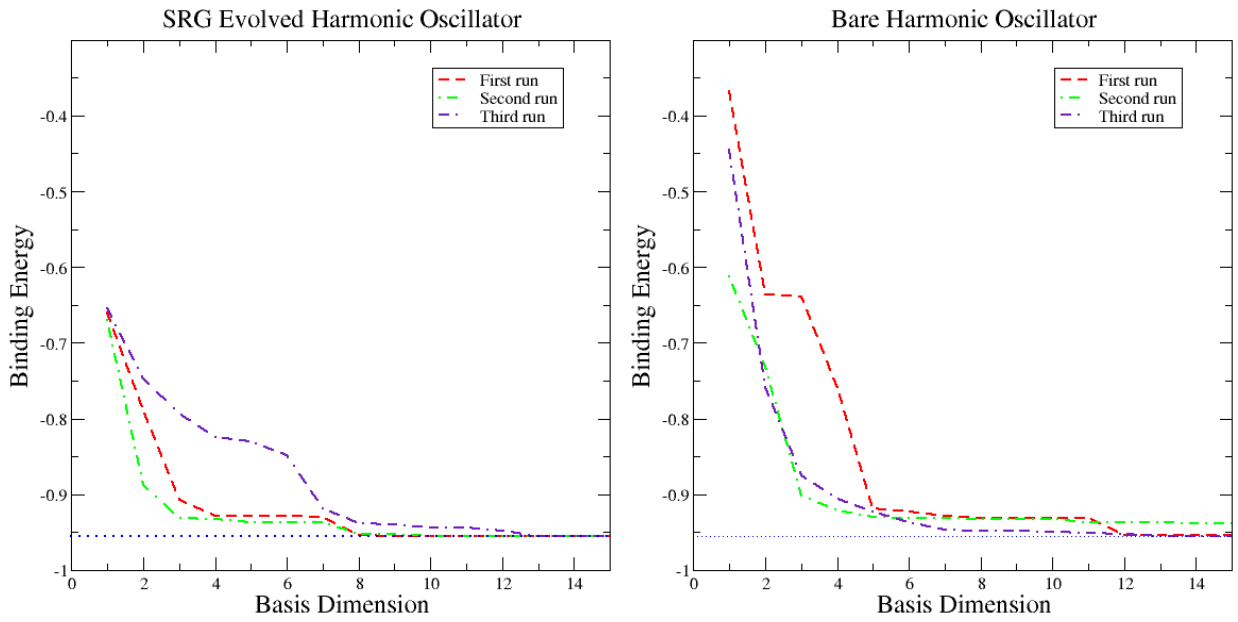

Figure 4: Calculated binding energy as basis functions are added for three independent constructions using the SHO with $g = 2.0$, $b = 0.030$, and $\lambda = 20.0$. The expected energy, -0.954714 in natural units, is labeled with a straight dotted line. Left: SRG evolved potential. Right: Bare potential.

the observables. By calculating the binding energy using only momenta that fall below the cutoff λ, Figure 5 demonstrates that this physical interpretation is accurate. The SRG potential $< \lambda$, shown on the left, not only still converges upon the expected binding energy, but it also appears to do so faster than when it was being fully used. The bare potential $< \lambda$, shown on the right, suffers from the missing pieces and converges upon the *wrong* binding energy in all three runs. This is because it is not as soft as the SRG potential and the low- to high-momentum coupling was not removed prior to discarding the high-momentum region. Missing these essential pieces of the potential, it fails to reach the expected binding energy. This confirms that the coupling effects are significant, that the low-momentum observables in an SRG potential are indeed unchanged, and that the SVM is effective on such potentials. An added bonus is that since the integrals within the matrix elements are evaluated numerically, doing them over such a restricted space dramatically decreases the time needed to run a calculation.

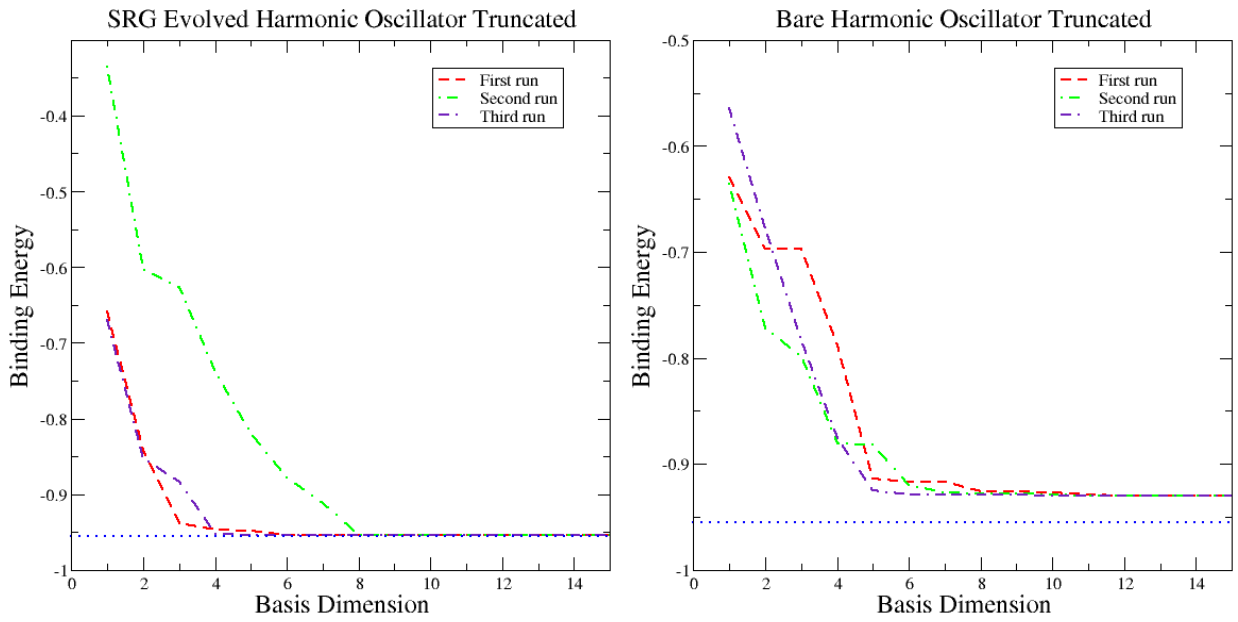

Figure 5: Calculated binding energy from only low momentum part of potential as basis functions are added for three independent constructions using the SHO with $g = 2.0$, $b = 0.030$, and $\lambda = 20.0$. The expected energy, -0.954714 in natural units, is labeled with a straight dotted line. Left: SRG evolved potential. Right: Bare potential.

A Few Words About The Code

The SVM code that I have written spans about 2400 lines in C++ over multiple files. This program utilizes a LAPACK routine for matrix diagonalizations and a file written by Prof. Dick Furnstahl for computing the points and weights needed for a gaussian quadrature. I have also written programs that perform linear and non-linear fits to given data. These programs are around 200 and 500 lines respectively. They both use GSL routines for computation of the fits.

The SVM code computes the hamiltonian and overlap matrix elements for the current basis of size N:

$$H_{ij} = \langle \psi(\alpha_i)|H|\psi(\alpha_j)\rangle \quad (i,j = 1,2,\ldots,N) \tag{12}$$

$$N_{ij} = \langle \psi(\alpha_i)|\psi(\alpha_j)\rangle \quad (i,j = 1,2,\ldots,N) \tag{13}$$

where the α parameters are those that appear in Equation 9. These matrices are passed to the LAPACK routine dsygv which solves the generalized eigenvalue equation $Hx = \lambda Nx$ and returns the eigenvalues of the system.

To build up a basis, we increase the basis size to $N + 1$ and stochastically minimize the binding energy through the new basis function $\psi(\alpha_{N+1})$. The minimization process requires that a pool of randomly generated parameters, within a given reasonable range, be generated and each one tested to see if it produces the lowest ground state energy. This means that the hamiltonian and overlap matrices are recalculated every time a new parameter is being tested.

The matrix elements can be computed using either a gaussian quadrature method or with an analytic form if one exists. For reasons of speed, it is obviously desired that the calculations be done analytically. This is the motivation for the development of the fitting functions discussed later.

The process for refining a pre-built basis is similar to the process of building it up. Instead of adding a new state, we pick a single α_i, holding all others constant, and minimize it from another pool of randomly generated parameters. These parameter pools are generated anew for every basis function we minimize in both the building and refinement processes.

The SRG evolved potentials come to me on a two-dimensional gaussian quadrature mesh from a MATLAB program written by Prof. Dick Furnstahl. These can be run directly into the SVM code or can be sent to the fitting routines in an attempt to fit the mesh to a sum of gaussian-type functions. The linear fitting routine has the most success with a sum of functions of the form

$$Y_i = A_i \, e^{\gamma_i (k + (-1)^i k')^2} \ . \tag{14}$$

To make this form suitable for a linear fit, the γ_i parameters are chosen ahead of time and are logarithmically distributed. This leaves only the A_i parameters, which appear linearly, to be fit to.

The SRG fits are weighted so as to emphasize the low-momentum parts. The weights are

$$w = \begin{cases} 1 & \text{if } k \text{ or } k' \text{ is} \leq 2\lambda \\ 0 & \text{if } k \text{ or } k' \text{ are} > 2\lambda \end{cases} . \tag{15}$$

This emphasizes the low momentum area of the potential and marginalizes the areas that correspond to momenta higher than 2λ. Technically, the low momentum observables should only depend on momenta below λ, but 2λ is used for safety.

Once the SRG potential has been fit to a sum of functions as given in Equation (14), the hamiltonian and overlap matrix elements no longer need to be calculated through the slow gaussian quadrature method. Now there are closed-form solutions for both which can be quickly computed. For a sum of fit functions with ϕ terms and a basis size of N, they are

$$H_{ij} = \frac{1}{\sqrt{2}\pi} \left(\frac{\alpha_i \alpha_j}{\alpha_i^2 + \alpha_j^2} \right)^{3/2} \delta_{ij} + \sum_k^{\phi} \left(\pi + 2 \cot^{-1} \left(\frac{2\gamma_k}{\Theta_{ijk}} \right) \right) \frac{1}{2\Theta_{ijk}} \tag{16}$$

and

$$N_{ij} = \left(\frac{(\alpha_i \alpha_j)^3}{2(\alpha_i^2 + \alpha_j^2)} \right)^{1/2} \tag{17}$$

where

$$\Theta_{ijk} = \sqrt{2\alpha_j^2 \gamma_k + \alpha_i^2 (\alpha_j^2 + 2\gamma_k)}. \tag{18}$$

Using these analytic solutions to compute the matrix elements dramatically increases the speed of the code, which will be invaluable when the method gets generalized to handle more bodies.

Discussion

It has been seen from both the MTV and harmonic oscillator potentials that the SVM works in momentum representation. It has also been demonstrated that the SVM is suitable for use on SRG evolved potentials. Knowing that the SVM works in our two-body test cases, it is appropriate to begin generalizing the method. The first reasonable problem to tackle is the one-dimensional triton (one proton and two neutrons). The problem is posed in a single dimension to avoid the complex angular interactions that occur between three or more particles. The appropriate trial wave function will no longer consist of a single width parameter, but will instead require a matrix of parameters that no longer directly correspond to widths. Similarly, there will be different relative momenta between each of the three particles, thus necessitating vectors of relative momenta. In general, the few-body trial wave function becomes [1]

$$\psi(A) = \exp\left(\frac{1}{2}\tilde{\mathbf{k}}A\mathbf{k}\right) = \exp\left(-\frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}A_{ij}\mathbf{x}_i\cdot\mathbf{x}_j\right) . \tag{19}$$

Another challenge associated with the triton and other few-body problems is that the trial wave functions are required to have the proper symmetry under exchange of identical particles. Bosons must be symmetric under this exchange and fermions must be antisymmetric. The procedure for permuting the trial wave functions would be easy if they were written in terms of single-particle coordinates, but the use of relative coordinates confounds the process. Our choice of gaussian wavefunctions is advantageous here because, relative to other basis choices, they can be more easily symmetrized.

For reasons of speed, it will also be prudent to optimize the minimization process. The current code diagonalizes the hamiltonian matrix for every stochastic basis parameter that gets tried. This cumbersome process can be circumvented with the use a root-solving algorithm. For instance, when constructing the basis, add the $K+1$'th basis to the *prediagonalized* hamiltonian of K states. The characteristic function of the new hamiltonian gives the equation

$$E - h_{K+1} = \sum_{i=1}^{K} \frac{|h_i|^2}{E - \epsilon_i} \ , \tag{20}$$

where

$$h_j = \langle \psi(\alpha_j) | H | \psi(\alpha_{K+1}) \rangle \tag{21}$$

and the ϵ_i's are the eigenvalues of the K-dimensional hamiltonian.[1] The roots E of this equation are the new eigenvalues of the $K+1$ hamiltonian. Thus the problem of calculating new eigenvalues reduces to finding the roots of Equation (20). This method can be similarly applied to the refinement process as well.

The program will require significant revisions to incorporate these new aspects for the triton. However, once these issues have been dealt with, the further addition of more nuclei will not present any radical changes to the existing code or technique. This project took the beginning steps towards a novel method for treating nuclear few-body problems. It inspires confidence that the SVM will work as advertised in more general few-body cases and how to proceed towards them.

Acknowledgements

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